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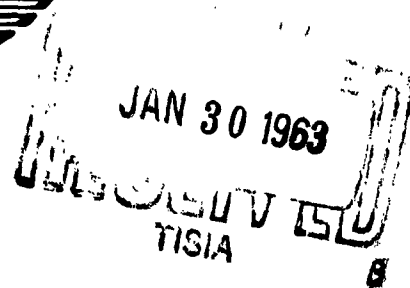
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ON GUIDANCE AND LANDING ACCURACY REQUIREMENTS IN RE-ENTRY TRAJECTORIES

by

Luigi Broglio



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ABSTRACT

The paper is divided into two parts.

In Part I. problems referring to guidance requirements are solved following the general approach of Ref.1; errors in velocity and angles, represented by means of errors in limiting conic parameters, produce variations in maximum deceleration total heat transferred angular ranges, etc. Charts to evaluate such variations are described.

In Part II perturbations with respect to the simplifying assumptions are considered and a general small-perturbations theory is developed. Any perturbation gives rise to changes in the limiting conic parameters: by considering the effect of a unit perturbation the effect of a distributed set of them is obtained by simple integration.

INTRODUCTION

Safe landing of a spacecraft requires, as well known, that some prescribed bounding values of heat transferred, maximum deceleration, range traveled, be not overcome. This can be achieved by prescribing an adequate program of lift and drag modulation: but - once this has been done - a major problem is a guidance problem, i.e., to give the proper velocity vector to the spacecraft at a given point.

Following a preceeding work of the Author (Ref 1), instead of the velocity vector, and without any need to define a "re-entry altitude", the parameters of the limiting, or approach conic can be selected. Thus, the problem arises to evaluate the effects on engineering quantities (such as heat, deceleration, range, etc.) of errors on such limiting conic parameters, or, conversely, to evaluate maximum tolerable errors in limiting conic parameters so as to prevent undue overcoming of engineering quantities boundaries. This can be done if a complete set of results, connecting engineering quantities to conic parameters is available; if a graphical presentation is prepared, very simple constructions allow a ready answer to the most of guidance requirements problems.

Some errors in the evaluation of such guidance problems can

arise from the simplified scheme upon which the theoretical analysis is performed. Secondary causes can sometimes produce noticeable effects, which are to be calculated. If the 'cause' is a first-order one a linear theory of small-perturbations can be developed. Disturbances arising from nonsphericity of the earth, variation of gravity, mass changes during flight (ablation) can be adequately represented by disturbing forces acting along the various points of the trajectory: and the basic singularity is then the effect of a unit pulse acting at a generic point of the trajectory. By using the approach of limiting conic parameters, the said disturbing pulse can be regarded as that producing a change in the limiting conic parameters of the portion of trajectory following it. And, so, each point of the disturbed trajectory is considered as belonging to a (fictitious) trajectory whose limiting conic is changed with respect to the undisturbed one; the change is given by the sum of those produced by the disturbances preceeding the point under concern.

PART I

GUIDANCE REQUIREMENTS FOR RE-ENTRY

1. - *Introductory remarks and recalls on similar solutions.*

It has been shown in Ref. 1 that re-entry trajectories of spacecrafts are described by the two simultaneous differential equations:

$$\left. \begin{aligned} r^2 \frac{d}{dr} \left(\frac{1}{r^2 \cos^2 \theta} \right) + \frac{h}{w^2} - \frac{L}{D} \frac{k\rho}{\cos^3 \theta} &= 0 \\ \frac{d \log \frac{h}{w^2}}{dr} + \frac{k\rho}{\sin \theta} \left(1 - \frac{L}{D} \tan \theta \right) &= 0 \end{aligned} \right\} \quad (1)$$

Here the independent variable r is the distance from the earth's center, whereas the unknowns are θ (flight path angle) and w (areal velocity = {velocity V } $\times r \cos \theta$); h is the gravitational constant (gravity acceleration $g \times 2r^2$), ρ is the density. The parameter k is equal to $(C_D A)/m$, (where C_D is the drag coefficient, A is the frontal area, m is the body mass), and can be modulated along the trajectory, as well as the lift/drag ratio (L/D). Modulation can also include variation of C_D vs (L/D) (Ref. 2).

Eqs. (1) were solved in Ref. (1) for the density law $\rho r^\alpha = \text{const.}$, ($\alpha = 900$, for the earth). It should be pointed out that such law is prac-

tically coincident with the commonly accepted one $\rho e^{\beta z} = \text{const.}$ (with $z = \text{altitude}$, $\beta = \frac{a}{\text{earth radius}}$) and lends itself much better for similarity purposes.

In order to obtain similarity laws it is interesting to take as reference values the quantities at the point where the total deceleration reaches its maximum value, or deceleration peak (DP). If there is more than one deceleration peak, reference can be made to any one of them. Quantities at DP (denoted by a star) are connected by the equation:

$$k_* \rho_* r_* = a_* \sin \theta_* \quad (2)$$

where: k_* is the value of k at DP; and:

$$a_* = \frac{a - \left\{ \frac{d}{d(r/r_*)} \left[\frac{c_D A}{m} \sqrt{1 + \left(\frac{L}{D} \right)^2} \right] \right\}_*}{1 - \frac{\sin \theta_*}{n_*} \sqrt{1 + \left(\frac{L}{D} \right)^2_*}} \quad (3)$$

Thus, if the non-dimensional abscissa is introduced:

$$\xi = 1 - \frac{r_*}{r} \quad (4)$$

and it is set:

$$k = k_* \eta(\xi) \quad ; \quad \frac{L}{D} = \lambda \varphi(\xi) \quad (5)$$

Eqs.(1) can be reduced, in nondimensional form, to the only

equation:

$$(1-\xi)^2 \frac{d}{d\xi} \log \left\{ \frac{\eta \lambda \varphi(\alpha_* \sin \theta_*) (1-\xi)^\alpha}{\cos^3 \theta} - \frac{d}{d\xi} \left(\frac{(1-\xi)^2}{\cos^2 \theta} \right) \right\} + \\ + \frac{(\alpha_* \sin \theta_*) \cdot \eta (1-\xi)^\alpha}{\sin \theta} (1 - \lambda \varphi \tan \theta) = 0 \quad (6)$$

The boundary conditions to Eqs. (6) can be established in two different ways. According to the first way, values of maximum deceleration $(n_* g_*)$ and angle θ_* can be prescribed at DP ($\xi = 0$). According to the alternative way, it is seen that the limiting form of Eqs. (1) - as ρ approaches zero - yield the two properties of the keplerian conic: areal velocity $w = \text{const.}$; total energy $E = V^2 - h/r = \text{const.}$ And so, the boundary conditions to Eqs. (6) may also prescribe the nondimensional limiting value of the conic in variants:

$$\text{as } \xi \rightarrow 1 \quad \left\{ \begin{array}{l} \lim \frac{w^2}{hr_*} = \kappa = \text{const.} \\ \lim \frac{V^2 - (h/r)}{h/r_*} = \epsilon = \text{const.} \end{array} \right. \quad (7)$$

Since, however, the two alternative ways of assigning boundary conditions to Eq. (6) must provide - obviously - the same solution - there must be obviously a correspondence of the type:

$$\left. \begin{array}{ll} \epsilon = \epsilon(n_*, \theta_*) & n_* = n_*(\epsilon, \kappa) \\ \kappa = \kappa(n_*, \theta_*) & \theta_* = \theta_*(\epsilon, \kappa) \end{array} \right\} \quad (8)$$

The problem is so established in a similar form (since the ballistic parameter is eliminated), with boundary conditions at the deceleration peak, and a continuous linkage to the outer space. For all quantities of interest for engineering purposes a similarity law will hold, and Ref. 1 provides the law of similarity for each of them.

2 - Limiting conic parameters

The limiting conic parameters chosen in Ref. 1 are, as said above, the total energy:

$$E = v^2 - \frac{h}{r} \quad (9)$$

and the areal velocity (or angular momentum):

$$w = V r \cos \theta \quad (10)$$

They can also be made nondimensional, and so the two independent parameters ϵ , κ can be defined:

$$\left. \begin{aligned} \epsilon &= \frac{E_{\infty}}{h/r_*} \\ \kappa &= \frac{w_{\infty}^2}{h r_*} \end{aligned} \right\} \quad (11)$$

The above choice, although correct in principle needs a reconsideration in view of guidance problems.

In fact, an adequate choice should lead to a pair of parameters which be sufficiently uncoupled, that is: one depending almost entirely on the velocity at a prescribed height, and the other almost entirely on the angle at the same height. In this way re-entry conditions are well described by two such parameters. Now it is easily seen that ϵ is depending only on V ; this is not the case with κ with respect to the angle θ .

The equation of nondimensional limiting conic may be written (Ref. 1):

$$1 + \tan^2 \theta = \frac{1 - \xi + \epsilon}{\kappa(1 - \xi)^2}$$

The value χ of $\tan^2 \theta$ at DP ($\xi = 0$) is consequently given by

$$\chi = \frac{1 + \epsilon}{\kappa} - 1 \quad (12)$$

It should be pointed out that χ is not the angle of the actual trajectory at DP (its value has been denoted by θ_*) nor is it necessarily a positive value: in fact, if the limiting conic perigee is higher than r_* , the angle χ is imaginary, and χ is negative.

In any case its value is sufficiently characteristic of the behavior of the limiting conic from the point of view of the an

gle θ , whereas ϵ is most adequately representing the velocity V . The parameters ϵ and χ will be chosen throughout this work for guidance analysis purposes.

3 - Guidance errors and corresponding variations of limiting conic.

Errors in guidance arise from a wrong estimation of angle θ , velocity V , at a given radius r .

Obviously, such errors correspond to undue variations in limiting conic parameters with respect to design values. Thus:

$$\left. \begin{aligned} \Delta E_{\infty} &= \Delta(V^2) \\ \Delta w_{\infty} &= r \Delta(V \cos \theta) \end{aligned} \right\} \quad (13)$$

whence also the errors in nondimensional parameters $\Delta\epsilon$ and $\Delta\bar{\chi}$ can be calculated:

$$\left. \begin{aligned} \Delta\epsilon &= \Delta\epsilon(\Delta V, \Delta\theta) \\ \Delta\bar{\chi} &= \Delta\chi(\Delta V, \Delta\theta) \end{aligned} \right\} \quad (14)$$

It is of particular interest to consider then case that quantities such as ΔV , $\Delta\theta$ may be regarded as relatively small changes about given values V and θ . It is thus possible to obtain $\Delta\epsilon$ and $\Delta\chi$ simply as:

$$\left. \begin{aligned} \Delta \epsilon &= \frac{\partial \epsilon}{\partial V} \Delta V + \frac{\partial \epsilon}{\partial \theta} \Delta \theta \\ \Delta \chi &= \frac{\partial \chi}{\partial V} \Delta V + \frac{\partial \chi}{\partial \theta} \Delta \theta \end{aligned} \right\} \quad (15)$$

The above written partial derivatives, according to Eqs. (9), (10), (11), have the expression:

$$\left. \begin{aligned} \frac{\partial \epsilon}{\partial V} &= \frac{r_*}{h} 2V \\ \frac{\partial \epsilon}{\partial \theta} &= 0 \\ \frac{\partial \chi}{\partial V} &= \left(\frac{r_*}{r} \right)^2 (1 + \tan^2 \theta) 2 \left(\frac{h}{r} - \frac{h}{r_*} \right) \frac{1}{V^3} \\ \frac{\partial \chi}{\partial \theta} &= \left(\frac{r_*}{r} \right)^2 \frac{2 \tan \theta}{\cos^2 \theta} \left[1 - \frac{(h/r) - (h/r_*)}{V^2} \right] \end{aligned} \right\} \quad (16)$$

The foregoing expressions can be transformed so as to depend on ϵ , χ and from ξ (which means that the dependency of guidance errors on limiting conic parameters is influenced - although slightly, as it will be seen now - by the altitude).

In view of the application to similar solutions, reference is here made to nondimensional quantities, such as C_V , (Ref. 1) given by:

$$C_V = \frac{2 r_*}{h} V \quad (17)$$

and so the following relationships are obtained

$$\begin{aligned}
 \frac{\partial \epsilon}{\partial C_v} &= 2 \sqrt{\epsilon + 1 - \xi} \\
 \frac{\partial \epsilon}{\partial \theta} &= 0 \\
 \frac{\partial \chi}{\partial C_v} &= - \frac{2^{\frac{1}{2}} \xi (\chi + 1)}{(\epsilon + 1)(\epsilon + 1 - \xi)^{\frac{1}{2}}} \\
 \frac{\partial \chi}{\partial \theta} &= 2(\chi + 1) \sqrt{\frac{2(\chi + 1)(\epsilon + 1 - \xi)}{(\epsilon + 1)(1 - \xi)^2}} - 1
 \end{aligned}
 \tag{18}$$

It is therefore easily shown that.

- (i) since $\frac{\partial \epsilon}{\partial \theta} = 0$, ϵ is insensitive to angle variations;
- (ii) since ξ has generally small values, χ is essentially depending on θ
- (iii) the above indicated partial derivatives are depending - although slightly - on the value of ξ , i.e., on the altitude at which the errors are considered

The above said conclusion hold also for finite errors, although the dependency is not so easily shown.

4 - Effects on trajectory quantities

In Ref 1, Part I, it is shown as said that entry conditions are represented by the nondimensional parameters of the limiting conic (in this paper the quantities ϵ and χ have

been selected as typical quantities). It is also shown that all the quantities related to the trajectory, such as total heat Q , heat rate q ; velocity V , time t , angular range φ , etc., can be represented in a "similar" form, i.e., as the product of a unit dimensional quantity (depending on the characteristics of a particular body being considered) by a nondimensional quantity (which is related only to ϵ and χ , i.e., independent on the body itself). Thus, f.i.,

$$Q = Q_1(\text{body}) \cdot C_Q(\epsilon, \chi) \quad (19)$$

A complete summary of quantities such as Q_1 , q_1 , V_1 , etc., and of the coefficients C_Q , C_q , C_V , etc., is given in Tab.I, Ref.1.

It is also shown in Ref 1. as said above, that an alternative approach to the problem is to prescribe ratio n_* of maximum deceleration (at DP, of course) to local gravity and angle θ_* at the same point; such values, can be used as initial conditions (for $\xi = 0$) and so the relationship between entry and DP conditions are simply given through the equations:

$$\left. \begin{aligned} n_* &= n_*(\epsilon, \chi) \\ \theta_* &= \theta_*(\epsilon, \chi) \end{aligned} \right\} \quad (20)$$

There exist no simpler means to establish Eqs.(20) than a step-

-by-step integration of Eq.(6). Approximate analytical formulas will be obtained later.

The question now arises in which practical way, the effects of the variations $\Delta\epsilon$, $\Delta\chi$ of the foregoing Art. on n_* , θ_* , C_Q , C_q , C_v , etc., can be evaluated.

For this purpose the following procedure can be used. On a ϵ , χ cartesian diagram the curves $n_* = \text{const}$, $\theta_* = \text{const}$, $C_Q = \text{const}$, etc., are plotted; so at every point (describing a particular entry conditions), the values of n_* , θ_* , C_Q , C_q , etc., can be read (Fig.1). (Hereafter the generic property is denoted by P , and its nondimensional symbol by C_P).

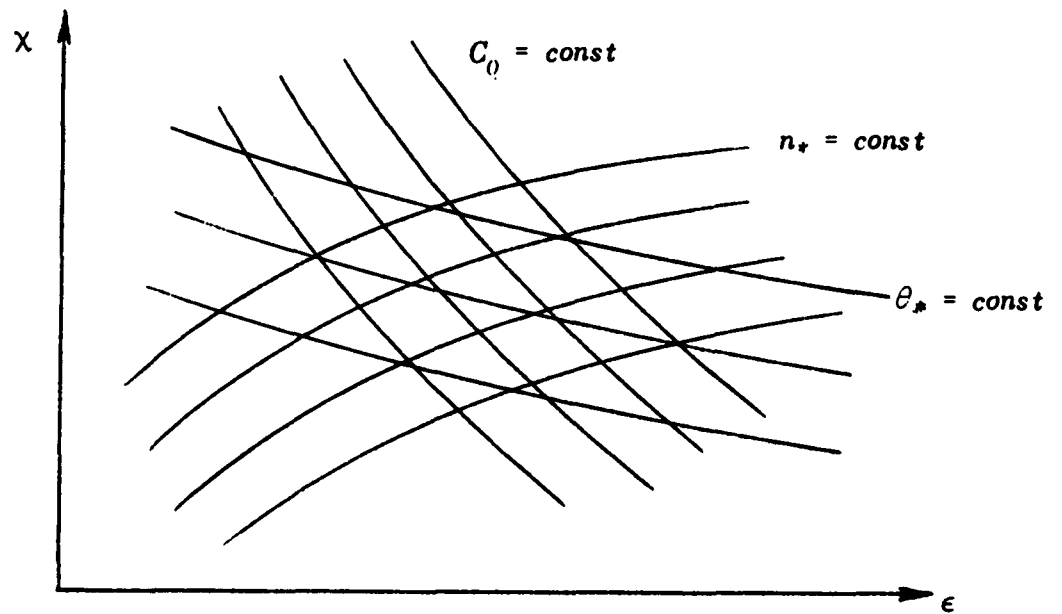


Fig. 1 - Charts for similar solutions.

When charts similar to that of Fig.1 are ready and available, any guidance problem can be solved:

- (i) Evaluation of effects such as the ΔC_p corresponding to prescribed variations $\Delta\epsilon$, $\Delta\chi$ is simply made by considering the point of coordinates $\epsilon + \Delta\epsilon$, $\chi + \Delta\chi$ and reading on the new points the changed quantity $C_p + \Delta C_p$ (Fig.2);

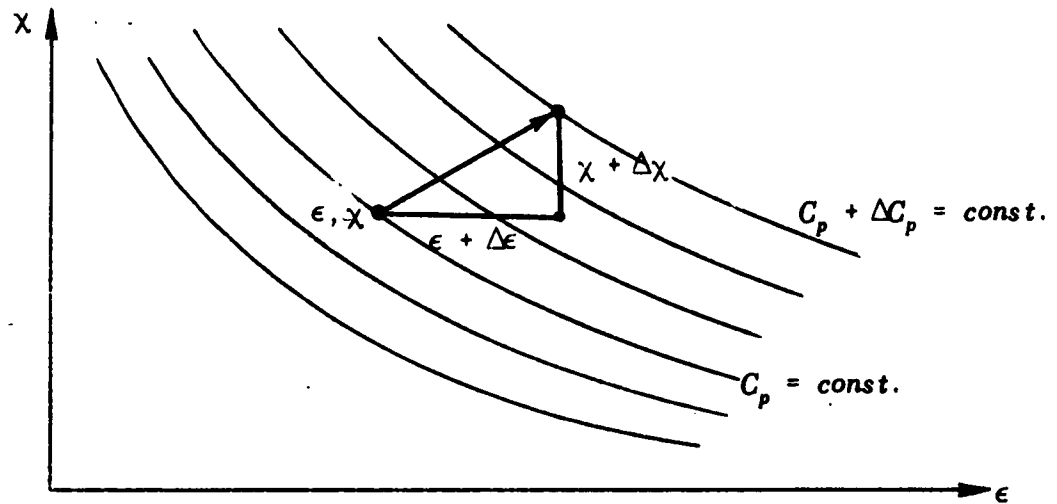


Fig. 2 - Effects of changes of limiting conic parameters.

- (ii) If one considers two quantities P_1 , P_2 (such as, f.i., C_Q and n_*), the question arises to find guidance requirements so as to allow maximum changes, about the design conditions, of value $+\Delta' P_1$; $-\Delta'' P_1$; $+\Delta' P_2$; $-\Delta'' P_2$ (Fig.3) In this case the shaded area describes the range of permitted variation of ϵ , χ about the starting point;
- (iii) It is also possible to obtain relationships connecting maximum allowable $\Delta\chi$ to a prescribed $\Delta\epsilon$ (and vice versa); this can be accomplished by associating to each $\Delta\chi$ (Fig.4) the corresponding value of $\Delta\epsilon$. If $\Delta\epsilon$ and $\Delta\chi$ are transformed into ΔC_v and $\Delta\theta$ as described in the previous

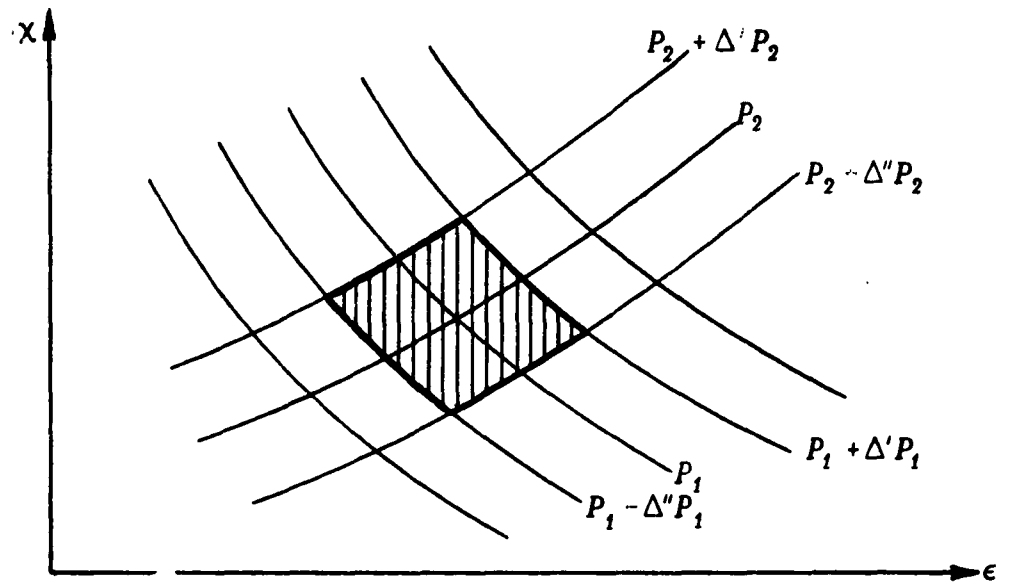


Fig. 3 - Evaluation of maximum allowable errors.

Article, a complete guidance requirement analysis can be performed.

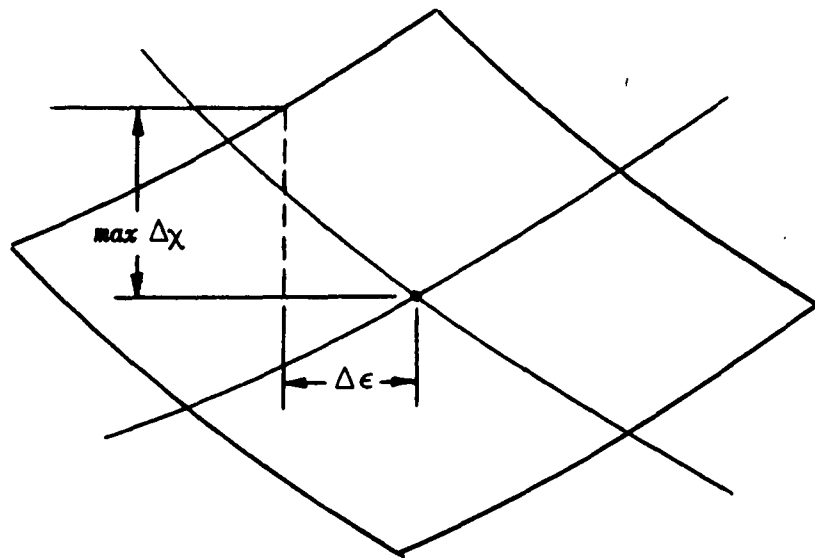


Fig. 4 - max $\Delta\chi$ for a prescribed $\Delta\epsilon$.

5 - Conic-DP charts

There appears to be a particular interest in quantities at DP (θ_* and n_*). The importance of n_* is stressed by its engineering meaning itself: whereas θ_* is important mainly in regard to the fact that it determines scale length (Eq. (2)).

Approximate relationship can be derived in this field. Their validity is enhanced by the fact that - in contrast to local approximate relationships - they are of average, or global, kind, and so are reasonably little influenced by the approximations themselves.

The general equations (6) are firstly re-written for sake of clearness.

$$\left. \begin{aligned} \frac{d \log (h/w^2)}{dr} + \frac{k\rho}{\sin \theta} \left(1 - \frac{L}{D} \tan \theta\right) &= 0 \\ r^2 \frac{d}{dr} \left(\frac{1}{r^2 \cos^2 \theta} \right) + \frac{h}{w^2} - \frac{L}{D} \frac{k\rho}{\cos^3 \theta} &= 0 \end{aligned} \right\} \quad (21)$$

(for symbols, see Glossary).

It is well known that, as ρ approaches zero, w^2 and E approach constant values, say w_∞^2 and E_∞ ; and the flight path angle θ_c follows the law:

$$\tan^2 \theta_c = \frac{E_\infty}{w_\infty^2} r^2 + \frac{h}{w_\infty^2} r - 1 \quad (22)$$

Introducing the nondimensional density:

$$x = \left(\frac{r_*}{r} \right)^\alpha \quad (23)$$

and letting:

$$\tan^2 \theta = \tan^2 \theta_c + \phi(x) \quad (24)$$

$\phi(x)$ must vanish at infinity ($x=0$), and must satisfy the differential equation:

$$\alpha x \frac{d\phi}{dx} + 2\phi = r \left(\frac{h}{w^2} - \frac{h}{w_\infty^2} \right) - \frac{L}{D} \varphi \frac{a_* \sin \theta_*}{\cos^3 \theta} x^{1-(1/\alpha)} \quad (25)$$

With a proper choice of the coefficients it is always possible to set:

$$\left. \begin{aligned} r \left(\frac{h}{w^2} - \frac{h}{w_\infty^2} \right) &= \sum_{n=0}^{\infty} a_n x^n \quad (a_0 = 0) \\ -\frac{L}{D} \varphi \frac{a_* \sin \theta_*}{\cos^3 \theta} &\simeq -\frac{L}{D} \varphi \frac{a_* \sin \theta_*}{\cos^3 \theta_*} = \sum_{n=0}^{\infty} b_n x^n \end{aligned} \right\} \quad (26)$$

and so, it is obtained from Eq.(25):

$$\begin{aligned} \phi &= \sum_{n=0}^{\infty} \left(\frac{a_n}{2 + \alpha n} x^n + \frac{b_n}{\alpha(n+1)+1} x^{n+1-(1/\alpha)} \right) \simeq \frac{1}{\alpha} \sum_{n=0}^{\infty} \left(\frac{a_n x^n}{n} + \right. \\ &\quad \left. + \frac{b_n}{n+1} x^{n+1-(1/\alpha)} \right) \end{aligned} \quad (27)$$

The problem is thus reduced to find the coefficients a_n and b_n . The first set of them (the a_n 's) is obtained approximately by stopping the expansion at $n=2$, and by imposing at DP the following conditions:

$$\left. \begin{aligned} w^2 &= w_*^2 \\ \frac{dw^2}{dr} &= \left(\frac{dw^2}{dr} \right)_* \\ \left(\frac{d^2 w^2}{dr^2} \right) &= \left(\frac{d^2 w^2}{dr^2} \right)_* \end{aligned} \right\} \quad \begin{array}{l} \text{as obtained from the equation of} \\ \text{motion and by the conditions at DP} \end{array}$$

Simple algebra (App. F) then provides the relationships:

$$\chi - \frac{11}{6\alpha} \frac{\chi + 1}{\epsilon + 1} = F_1 \quad (28)$$

$$\begin{aligned} & - \log \frac{\chi + 1}{\epsilon + 1} - \\ & - \frac{1}{3} \frac{\sin \theta_*}{\sqrt{\frac{\chi + 1}{\epsilon + 1} \frac{\epsilon + 1 - \xi_*}{(1 - \xi_*)^2} - 1}} = F_2 \end{aligned} \quad (29)$$

where:

$$\left. \begin{aligned} F_1 &= F_1(\theta_*, n_*) \\ F_2 &= F_2(\theta_*, n_*) \end{aligned} \right\} \quad (30)$$

are functions of quantities at DP, given in Appendix. ξ_e is the point at which density is sensibly zero. The system (28) (29) provides approximate conic-DP relationships.

6 - Typical Results

Fig.5 shows the conic-DP charts for an unmodulated re-entry with lift. The cases $L/D = 0$; $L/D = 0.1$; $L/D = 0.2$; $L/D = 0.5$ were considered.

It is seen that curves $n_* = \text{const.}$ can cut each other: this simply means that the same entry condition is corresponding to more than one deceleration peak. In the same plane it is also very easy to determine the kind of limiting conic (according to

the sign of ϵ). On the same plot curves such as skipping limit, re-exit on a circular orbit, homeless re-exit into space may also shown. (Here only the $L/D = 0$ case is presented)

Similar results are obtained for other values of L/D : it is seen however that increase of L/D gives rise to a softer trend in the above said curves. Furthermore no cases of more than one DP have been found.

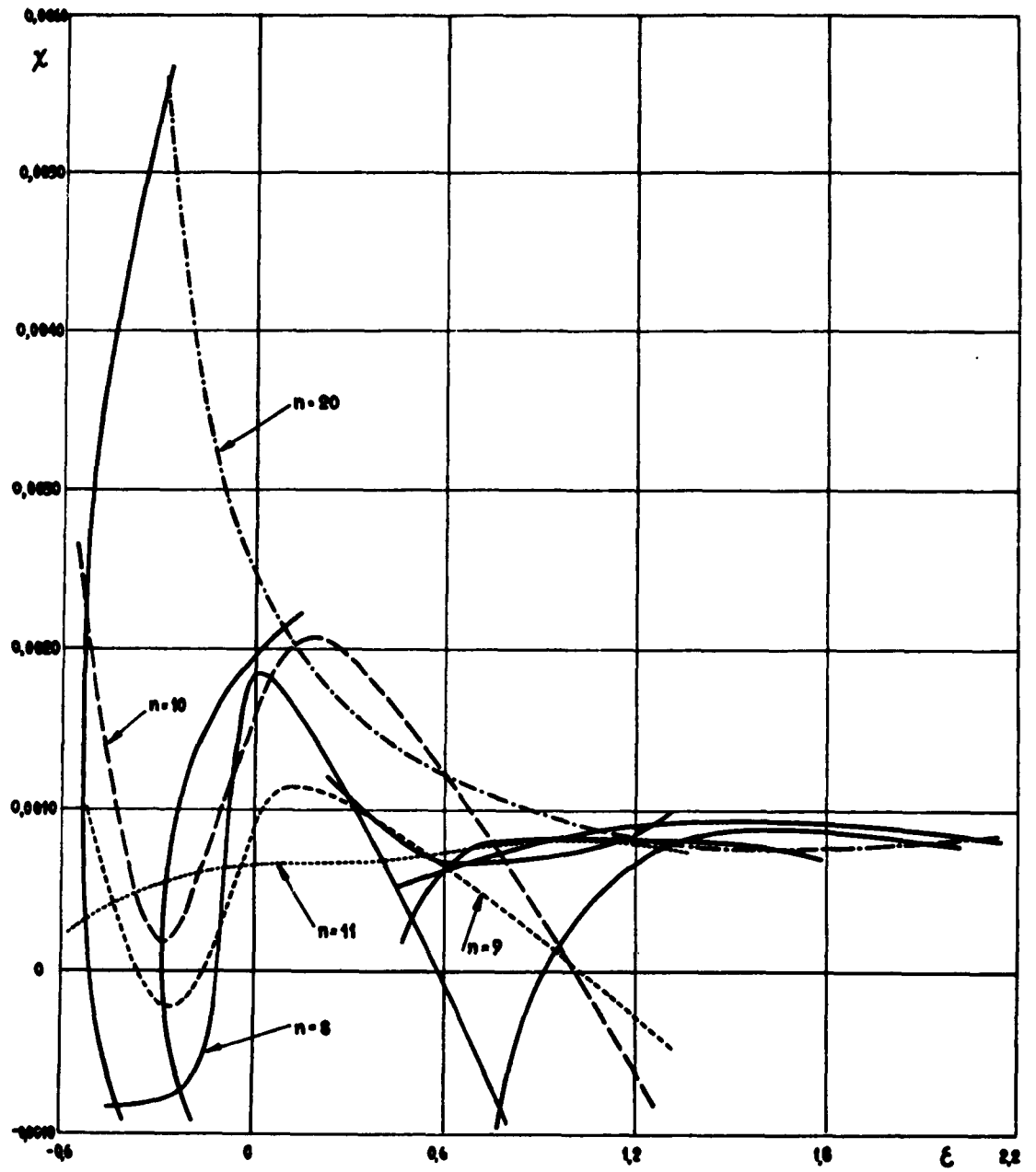


Fig. 5
CONIC - DP RELATIONSHIPS $(\frac{L}{D}) = 0$

PART II

SMALL-PERTURBATIONS THEORY IN RE-ENTRY TRAJECTORIES

1 - General considerations

The analytical model which was hitherto considered is a simplified one, i.e., a nonrotating radially symmetric planet and a radially symmetric atmosphere where the density is varying according to a prescribed law.

A more realistic picture of the phenomena must be however be considered sometimes. The effects of the simplifying assumptions can be evaluated by considering the effects of such assumptions as disturbances superimposed to the base-trajectory. Likewise, maneuvers during re-entry, in order to improve the accuracy of a safe landing can also be studied as perturbations superimposed to the base-trajectory.

The following Articles are devoted to evaluate the effects of a generic perturbation. It is assumed that such perturbation be a first-order one, so that a linearized theory can be applied.

The theory developed in Ref. 1 is based on Eqs.(1). They can be resolved along the tangent \underline{t} to the trajectory and along its normal \underline{n} in terms of accelerations. They can be written in sym-

bolic form:

$$\left. \begin{aligned} [a_t] &= 0 \\ [a_n] &= 0 \end{aligned} \right\} \quad (31)$$

whereas on the bi-normal the equation provides identically $0=0$.

A disturbing acceleration of components Δa_t , Δa_n , Δa_b is now considered, small enough to allow to discard changes in terms such as $[a_t]$ $[a_n]$. The equation of motion is consequently written as:

$$\left. \begin{aligned} [a_t] + \Delta a_t &= 0 \\ [a_n] + \Delta a_n &= 0 \\ \Delta a_b &= 0 \end{aligned} \right\} \quad (32)$$

If, as said above, first-order effects are considered, the evaluation of the effects due to Δa_t , Δa_n , Δa_b , can be made by considering a "unit" solution, and then superimposing the effects of the various perturbations.

2 - Unit perturbation

A generic point P (defined by its distance r from the planet center) of a re-entry trajectory is considered (Fig.6); the perturbation acting there is re-solved along the three lines (tangent, normal, bi-normal). The effect of Δa_t is an abrupt

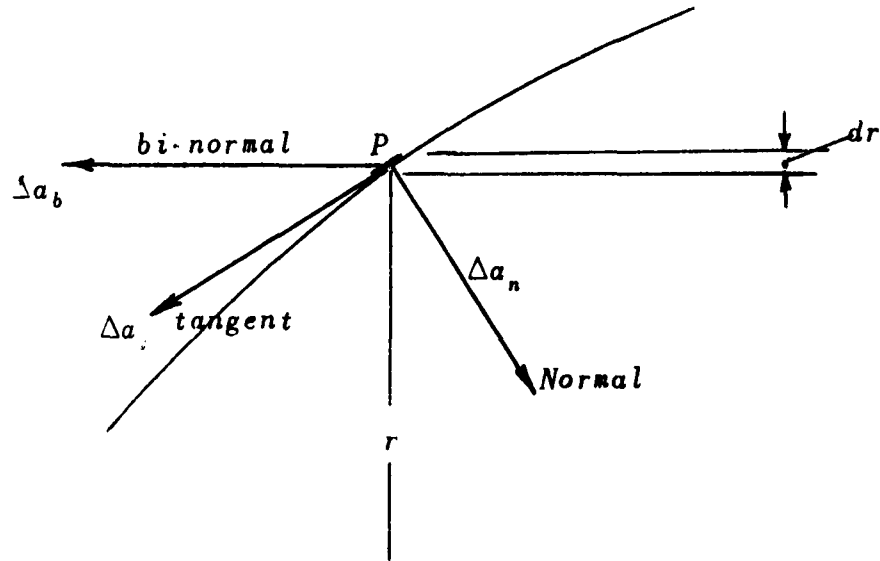


Fig. 6 - Unit perturbation

change in the total velocity given by:

$$dV = \Delta a_t \cdot dt \quad (33)$$

(where dt is the time elapsed to travel the distance dr). The effect of Δa_n is an abrupt change in flight path:

$$d\theta = \frac{\Delta a_n}{V} dt \quad (34)$$

The effect of Δa_b is a rotation of the plane of the trajectory.

$$d\psi = \frac{\Delta a_b}{V} dt \quad (35)$$

Since $dt = dr/(V \sin \theta)$ the foregoing equations can be written:

$$\left. \begin{aligned}
 dV &= \frac{\Delta a_t}{V \sin \theta} dr \\
 d\theta &= \frac{\Delta a_n}{V^2 \sin \theta} dr \\
 d\psi &= \frac{\Delta a_b dr}{V^2 \sin \theta}
 \end{aligned} \right\} \quad (36)$$

Since no perturbation are acting after P , the equations of the disturbed trajectory after such point are the same of the undisturbed one; with the conditions that velocity, flight path angle, and plane rotation at P are $V + dV$, $\theta + d\theta$, $d\psi$.

This is equivalent to say that the portion of trajectory following the point of perturbation is an indisturbed trajectory such that:

- (a) the plane is rotated of $d\psi$;
- (b) the limiting conic parameters are changed of such quantities $d\epsilon$, $d\chi$ that the variations dV , $d\theta$ are produced at P .

In other words, the solution corresponding to the perturbation is the one indicated in Fig. 7. Before r , the equations of the trajectory are the undisturbed ones and (ϵ, χ) are the limiting conic parameters; after r the equations are still the undisturbed ones, and $(\epsilon + d\epsilon, \chi + d\chi)$ are the limiting conic parameters. Besides that, the plane of the trajectory is rotated of an angle $d\psi$ about the focal axis of the limiting conic.

Naturally the question arises as how to relate $d\epsilon$, $d\chi$ to dV , $d\theta$, and vice-versa. This is very simply done by calculating

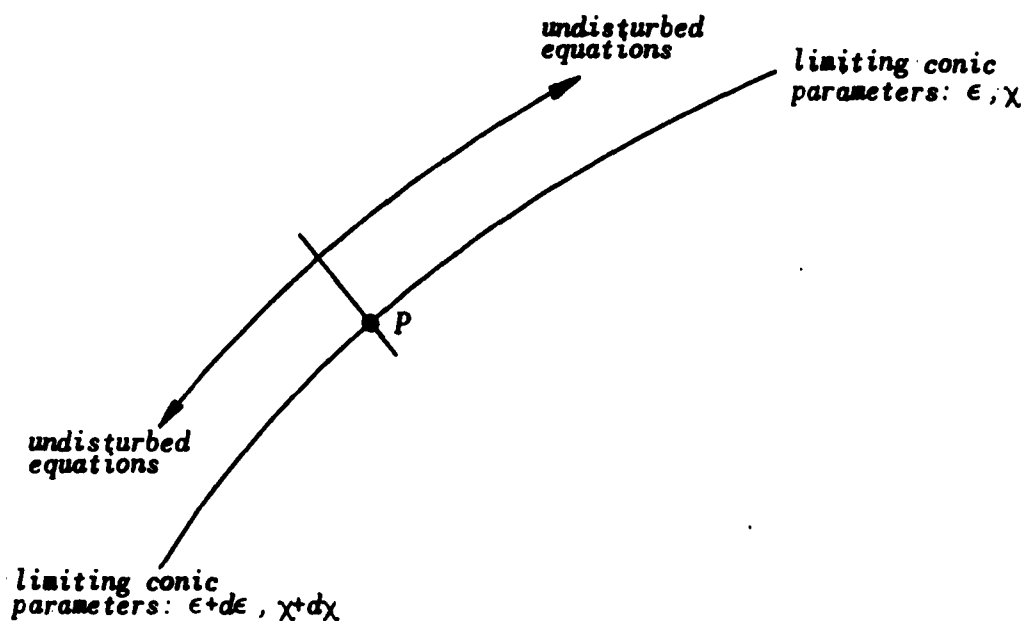


Fig. 7 - Change in limiting conic parameters.

three trajectories: one corresponding to (ϵ, χ) ; one corresponding to $(\epsilon + \delta\epsilon, \chi)$; one to $(\epsilon, \chi + \delta\chi)$. If $\delta\epsilon$, $\delta\chi$ are sufficiently small, by comparing quantities corresponding to the same ξ , one has, f.i.;

$$\frac{\partial(V/V_*)}{\partial\epsilon} \cong \frac{(V/V_*)_{\epsilon+\delta\epsilon, \chi} - (V/V_*)_{\epsilon, \chi}}{\delta\epsilon} \quad (37)$$

and so for all the other derivatives, such as:

$$\frac{\partial(V/V_*)}{\partial\chi}, \quad \frac{\partial\theta}{\partial\epsilon}, \quad \frac{\partial\theta}{\partial\chi}$$

whence, by simple inversion, also partial derivatives such as:

$\frac{\partial \epsilon}{\partial(V/V_*)}$; $\frac{\partial \epsilon}{\partial \theta}$; $\frac{\partial \chi}{\partial(V/V_*)}$; $\frac{\partial \chi}{\partial \theta}$ are evaluated.

Thus, coming back to the unit solution, the limiting conic of the points following p has parameters $\epsilon + d\epsilon$, $\chi + d\chi$ where:

$$\left. \begin{aligned} d\epsilon &= \frac{\partial \epsilon}{\partial(V/V_*)} \frac{\Delta a_t}{VV_* \sin \theta} dr + \frac{\partial \epsilon}{\partial \theta} \frac{\Delta a_n}{V^2 \sin \theta} dr \\ d\chi &= \frac{\partial \chi}{\partial(V/V_*)} \frac{\Delta a_t}{VV_* \sin \theta} dr + \frac{\partial \chi}{\partial \theta} \frac{\Delta a_n}{V^2 \sin \theta} dr \end{aligned} \right\} \quad (38)$$

and a rotation $d\psi$ of its plane:

$$d\psi = \frac{\Delta a_b dr}{V^2 \sin \theta} \quad (39)$$

3 - Distributed perturbations

If a continuous distribution Δa_t , Δa_n , Δa_b of perturbations is applied to the trajectory, each element of it can be considered as a portion of a trajectory having as limiting conic $\epsilon + \Delta \epsilon$, $\chi + \Delta \chi$ where $\Delta \epsilon$ and $\Delta \chi$ are the sum of the quantities $d\epsilon$, $d\chi$ preceeding the point under concern. And, so:

$$\left. \begin{aligned} \Delta \epsilon &= \int_{inf.}^r \left(\frac{\partial \epsilon}{\partial(V/V_*)} \frac{\Delta a_t}{VV_* \sin \theta} + \frac{\partial \epsilon}{\partial \theta} \frac{\Delta a_n}{V^2 \sin \theta} \right) dr \\ \Delta \chi &= \int_{inf.}^r \left(\frac{\partial \chi}{\partial(V/V_*)} \frac{\Delta a_t}{VV_* \sin \theta} + \frac{\partial \chi}{\partial \theta} \frac{\Delta a_n}{V^2 \sin \theta} \right) dr \\ \Delta \psi &= \int_{inf.}^r \frac{\Delta a_b dr}{V^2 \sin \theta} \end{aligned} \right\} \quad (40)$$

which is the general answer to the small perturbations theory in re-entry trajectories. This means that each element can be thought as that of limiting conic $\Delta\epsilon$, $\Delta\chi$. To calculate $\Delta\epsilon$ and $\Delta\chi$ it is necessary to know:

- (i) the four partial derivatives $\frac{\partial\epsilon}{\partial(V/V_*)}$, etc.
- (ii) the base trajectory;
- (iii) and - of course - the disturbances.

It is also seen that the small-perturbations theory can be applied to similar solutions

Indeed Eqs (40) can be written.

$$\begin{aligned}\Delta\epsilon &= \int_1^\xi \left\{ \frac{\partial r}{\partial(V/V_*)} \frac{r_* \Delta a_t}{V_*^2} \frac{1}{\sin\theta} + \frac{\partial\epsilon}{\partial\theta} \frac{r_* \Delta a_n}{V_*^2} \frac{1}{(V/V_*)^2 \sin\theta} \right\} \frac{d\xi}{(1-\xi)^2} \\ \Delta\chi &= \int_1^\xi \left\{ \frac{\partial\chi}{\partial(V/V_*)} \frac{r_* \Delta a_t}{V_*^2} \frac{1}{\sin\theta} + \frac{\partial\chi}{\partial\theta} \frac{r_* \Delta a_n}{V_*^2} \frac{1}{(V/V_*)^2 \sin\theta} \right\} \frac{d\xi}{(1-\xi)^2} \\ \Delta\psi &= \int_1^\xi \frac{r_* \Delta a_b}{V_*^2} \frac{1}{(V/V_*)^2 \sin\theta} \frac{d\xi}{1-\xi^2}\end{aligned}\quad (41)$$

where only similar quantities are appearing. Obviously new parameters must be considered which arise from the actual expressions of quantities such as Δa_t , Δa_n , Δa_b .

Once the quantities $\Delta\epsilon$, $\Delta\chi$, $\Delta\psi$ have been calculated, the perturbed velocities and angles at every point of the trajectory are given by

$$\Delta\theta = \frac{\partial\theta}{\partial\epsilon} \Delta\epsilon + \frac{\partial\theta}{\partial\chi} \Delta\chi$$

$$\Delta(V/V_*) = \frac{\partial(V/V_*)}{\partial\epsilon} \Delta\epsilon + \frac{\partial(V/V_*)}{\partial\chi} \Delta\chi$$
(42)

4 - Application to non-conductive ablation

An application of the above theory can be made to the case of bodies re-entry with a limited amount of ablation. The difference from the non ablating case is now in the circumstance in the fact that the mass m is no longer constant, but is varying along the trajectory according to the law:

$$\frac{dm}{dt} = - \frac{q}{K}$$
(43)

where K denotes the ablation coefficient (Ref.4).

If K is a constant:

$$m(\xi) = m_0 - \frac{Q(\xi)}{K}$$
(44)

where $Q(\xi)$ is the total heat transferred to the body from the beginning up to the point being considered.

The aforesaid mass variation can be considered, for every point of the trajectory, as a perturbation of components.

$$\begin{aligned}
\Delta a_n = \Delta a_b = 0 \quad ; \quad \Delta a_t &= \frac{1}{2} c_D \rho A V^2 \left(\frac{1}{m} - \frac{1}{m_0} \right) = \\
&= - \frac{\frac{1}{2} c_D \rho A V^2 (m - m_0)}{m_0^2} = \\
&= - \frac{1}{2} k \rho V^2 \frac{m - m_0}{m_0}
\end{aligned} \tag{45}$$

and so:

$$\Delta a_t = - \frac{1}{2} \frac{k \rho V^2 Q}{K m_0} \tag{46}$$

Relative ablation corresponding to the total heat Q_f transferred to the body up to landing is given by: (with \bar{m} = final mass):

$$\frac{\bar{m} - m_0}{m_0} = - \frac{Q_f}{K m_0} \tag{47}$$

and so:

$$\Delta a_t = \frac{1}{2} k \rho V^2 \frac{\bar{m} - m_0}{m_0} \frac{Q}{Q_f} = \frac{1}{2} k \rho V_*^2 \frac{\rho}{\rho_*} \left(\frac{V}{V_*} \right)^2 \frac{\bar{m} - m_0}{m_0} \frac{Q}{Q_f} \tag{48}$$

Therefore Eqs.(41) of the foregoing Article provide:

$$\left. \begin{aligned}
 \frac{\Delta \epsilon}{\frac{\bar{m} - m_0}{m_0}} &= \int_1^\xi \frac{a_* \sin \theta_*}{2} \left(\frac{\rho}{\rho_*} \right) \left(\frac{V}{V_*} \right)^2 \frac{\partial \epsilon}{\partial (V/V_*)} \frac{d\xi}{(1-\xi)^2 \sin \theta} \\
 \frac{\Delta \chi}{\frac{\bar{m} - m_u}{m_0}} &= \int_1^\xi \frac{a_* \sin \theta_*}{2} \left(\frac{\rho}{\rho_*} \right) \left(\frac{V}{V_*} \right)^2 \frac{\partial \chi}{\partial (V/V_*)} \frac{d\xi}{(1-\xi)^2 \sin \theta} \\
 \Delta \psi &= 0
 \end{aligned} \right\} (49)$$

which express the variations of limiting conic parameters corresponding to a total ablation ($\bar{m} = 0$). For $\bar{m} \neq 0$ it is enough to multiply the corresponding quantities by the ablation rate.

5 - Numerical example

A numerical case was calculated with the data of Fig. 8. In the same plot the velocity variation *vs.* ξ is represented for a trajectory having a circle as limiting conic. The values of ΔV refer to the case of complete ablation: in the case of partial ablation it is enough - as said above - to multiply the values by the percentage of ablated weight. Thus, f.i., at DP ($\frac{V}{V_*} = 1$), the ordinate of the plot is 0.24, and, for an ablation of 30%, one would have $\frac{\Delta V}{V_*} \simeq 0.08$. Similar considerations apply to the case of escape velocity (Fig. 9). In this case the change of velocity for an ablation of 30% is $\simeq 0.07$. Both cases refer to a maximum deceleration of 10g

Fig. 10 and Fig. 11 show respectively the flight path angle variation for a complete ablation.

In this case the effect is to increase the angle. For the orbital speed and ablation of 30% gives a change of the angle of about $1/8$ of the unperturbed value; the corresponding value for the escape velocity is about 8%

VARIATION OF VELOCITY FOR TOTAL ABLATION
 $\xi = -0.49333$, $\chi = 0.0025724$; $\frac{L}{D} = 0$
 (CIRCULAR RE-ENTRY WITH $n_n = 10$)

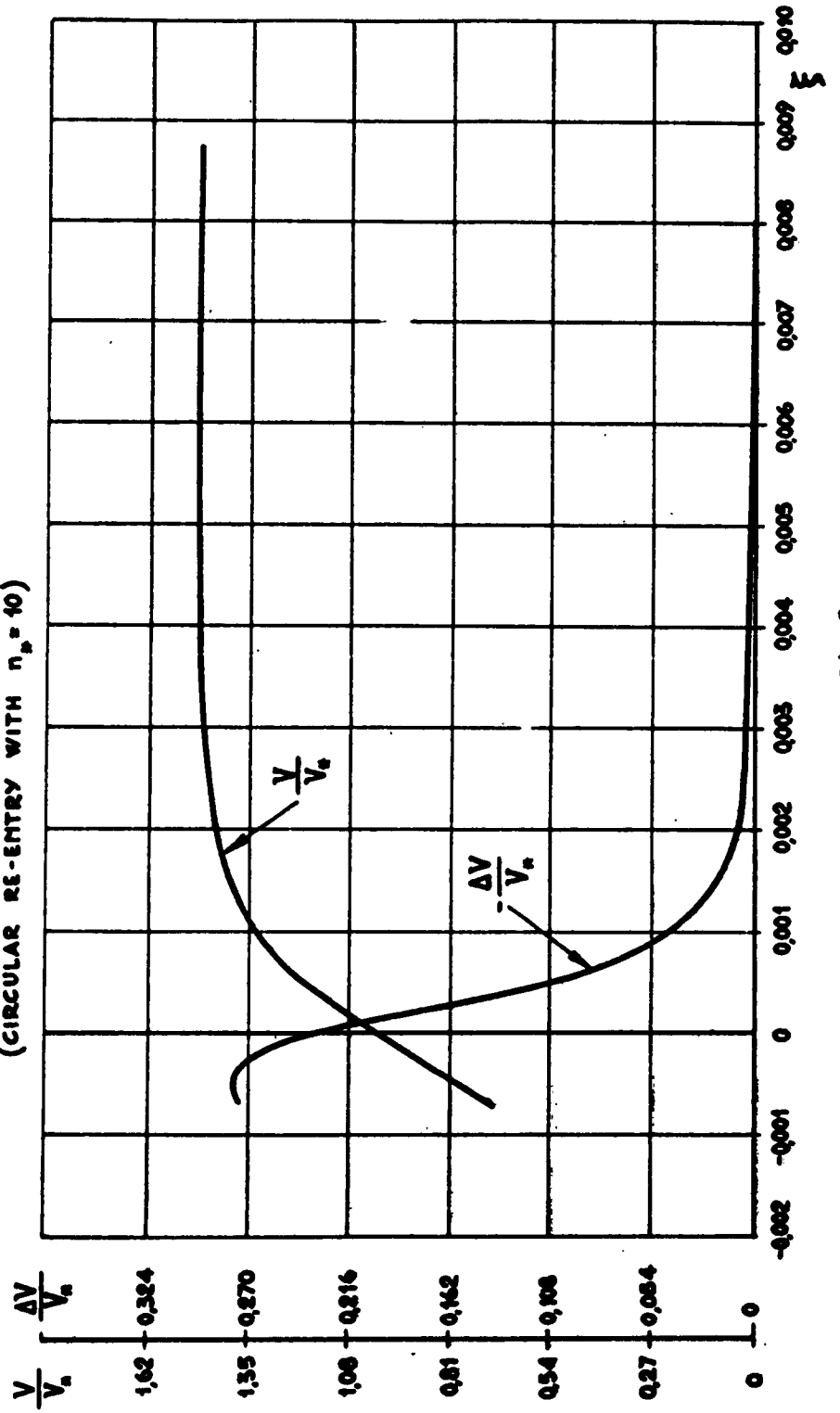


Fig. 8

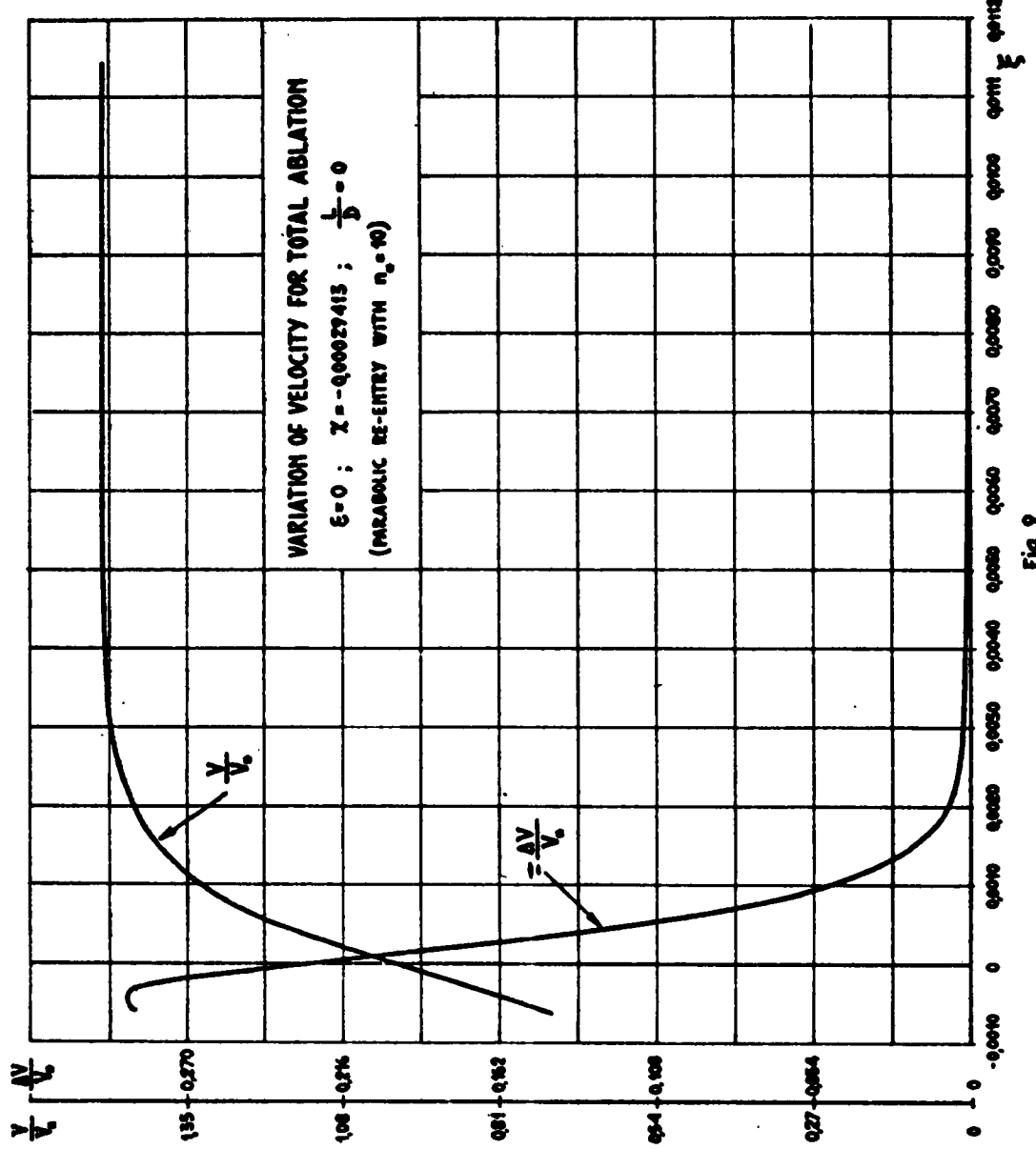


Fig. 9

VARIATION OF FLIGHT PATH ANGLE FOR TOTAL ABLATION

$\xi = -0.49333$; $\chi = -0.0025724$; $\frac{L}{D} = 0$
(CIRCULAR RE-ENTRY WITH $n_0 = 10$)

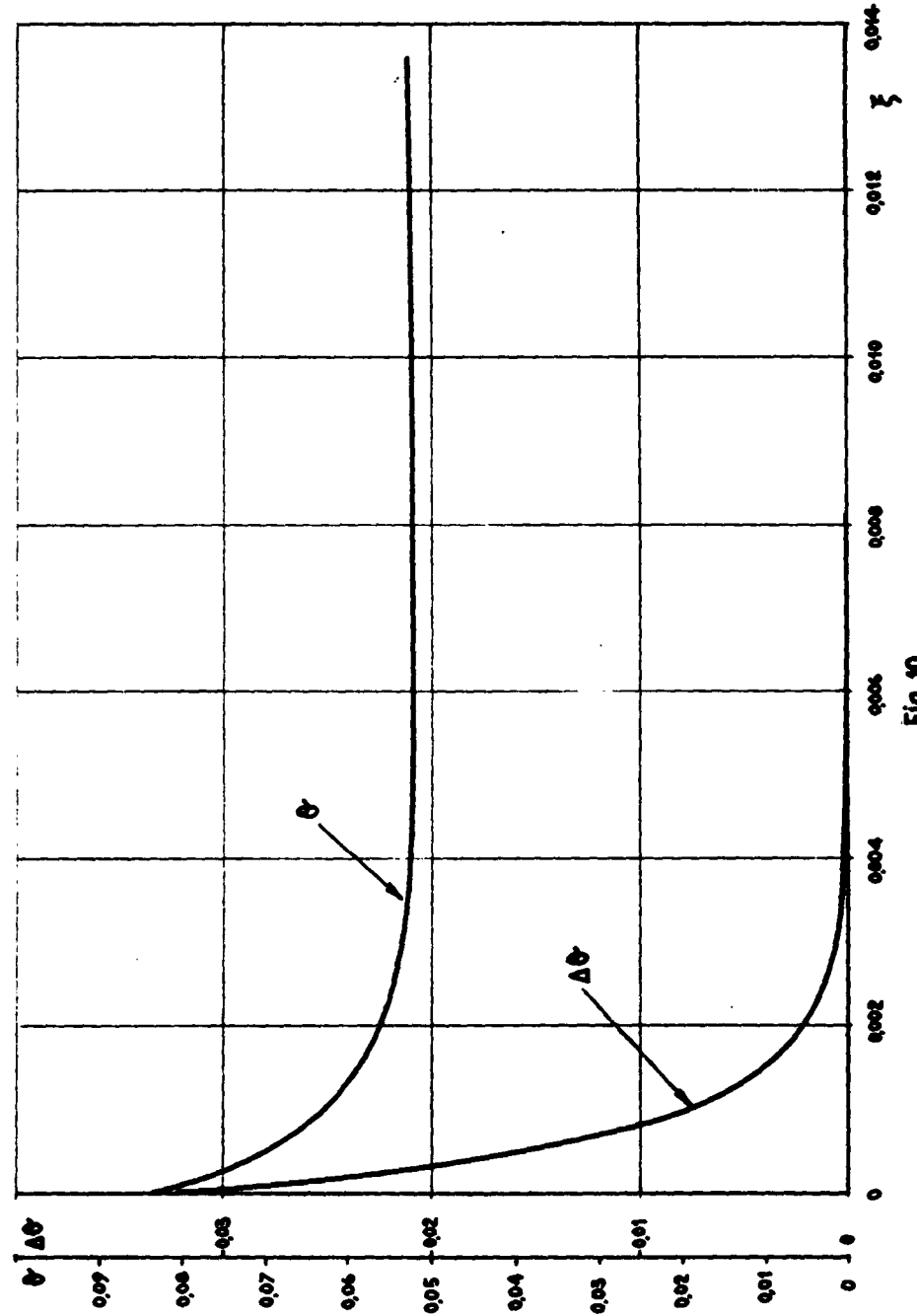


Fig. 10

VARIATION OF FLIGHT PATH ANGLE FOR TOTAL ABLATION

$\epsilon = 0$; $\chi = -0.00029413$; $\frac{L}{D} = 0$
(PARABOLIC RE-ENTRY WITH $n_p = 10$)

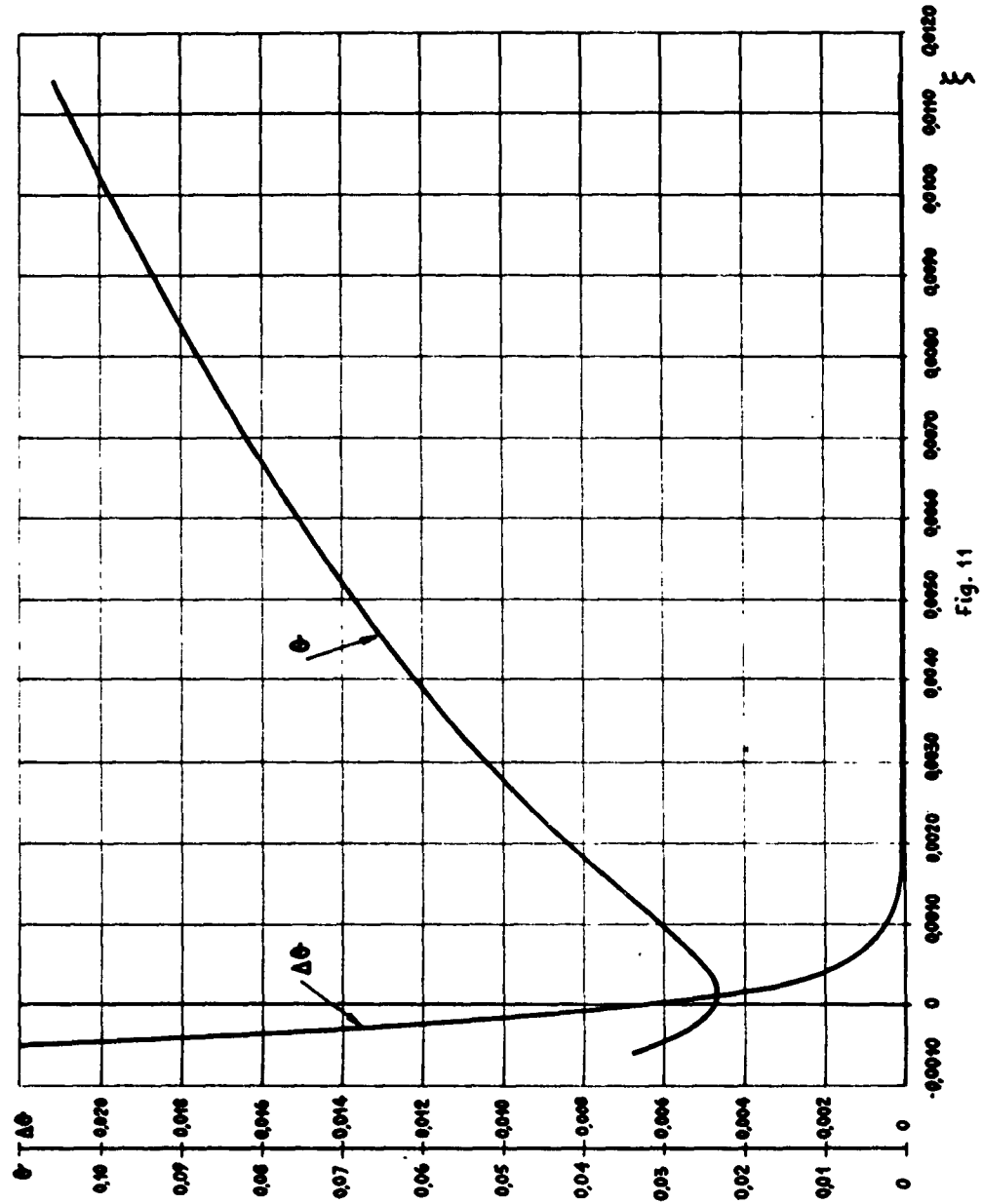


Fig. 11

A P P E N D I X

DETERMINATION OF APPROXIMATE CONIC - DP RELATIONSHIPS

1. Let:

$$z = \frac{hr_*}{w^2} \quad (A1)$$

Remembering (23), the first of Eqs.(26) is written.

$$x \cdot \frac{1}{a} (z - z_\infty) = a_1 x + a_2 x^2 + a_3 x^3 \quad (A2)$$

Denoting by a dot derivatives with respect to x , the conditions on a_1, a_2, a_3 for identifying Eqs.(A2) and its first and second derivative at DP, are:

$$z_* = a_1 + a_2 + a_3 + z_\infty$$

$$\dot{z}_* - \frac{1}{a} z_* = a_1 + 2a_2 + 3a_3$$

$$\ddot{z}_* - \frac{2}{a} \dot{z}_* + \frac{1}{a} \left(\frac{1}{a} + 1 \right) z_* = 2a_2 + 6a_3$$

whence.

$$a_1 = \frac{\ddot{z}_*}{2} - \left(2 + \frac{1}{a}\right) \dot{z}_* + \left\{\left(1 + \frac{1}{a}\right)\left(2 + \frac{1}{a}\right) + 1\right\}(z_* - z_\infty)$$

$$a_2 = -\ddot{z}_* + \left(3 + \frac{2}{a}\right) \dot{z}_* - \left\{\left(1 + \frac{1}{a}\right)\left(3 + \frac{1}{a}\right)\right\}(z_* - z_\infty)$$

$$a_3 = \frac{\ddot{z}_*}{2} - \left(1 + \frac{1}{a}\right) \dot{z}_* + \left\{\left(1 + \frac{1}{a}\right)\left(2 + \frac{1}{a}\right)\right\} \frac{(z_* - z_\infty)}{2}$$

If quantities of the order of $1/a$ are neglected as compared to unity, one has finally:

$$\left. \begin{aligned} a_1 &= \frac{1}{2} \ddot{z}_* - 2\dot{z}_* + 3(z_* - z_\infty) \\ a_2 &= -\ddot{z}_* + 3\dot{z}_* - 3(z_* - z_\infty) \\ a_3 &= \frac{\ddot{z}_*}{2} - \dot{z}_* + (z_* - z_\infty) \end{aligned} \right\} \quad (A3)$$

2. Evaluation of \dot{z}_* , \ddot{z}_* is now performed.

Form the second of Eqs. (1), remembering Eqs. (23), (2), (5), of Part I:

$$\frac{dz}{dx} = \frac{\alpha_* \sin_* \theta}{\alpha} \frac{z \eta x^{-1/a}}{\sin \theta} [1 - \lambda \varphi \tan \theta]$$

whence:

$$\dot{z}_* = \frac{a_*}{a} z_* [1 - \lambda \varphi_* \tan \theta_*] \quad (\text{A4})$$

The second derivative is now performed. It is obtained:

$$\begin{aligned} \ddot{z} = & \frac{a_* \sin \theta_*}{a} [z \eta x^{-1/a} \{ -(\frac{\cos \theta}{\sin^2 \theta} + \lambda \varphi \frac{\sin \theta}{\cos^2 \theta}) \dot{\theta} - \frac{\lambda \dot{\varphi}}{\cos \theta} \} + \\ & + \frac{(1 - \lambda \varphi \tan \theta)}{\sin \theta} (\dot{z} \eta x^{-1/a} + z \dot{\eta} x^{-1/a} - \frac{1}{a} z \eta x^{-\frac{1}{a}-1})] \end{aligned}$$

The value of θ at DP is obtained from the first of Eq. (), and it is easily obtained

$$\dot{\theta}_* = -\frac{\cot \theta_*}{a} + \frac{z_* \cos^3 \theta_*}{2a \sin \theta_*} - \frac{\lambda \varphi_* a_*}{2a}$$

and so:

$$\begin{aligned} \ddot{z}_* = & z_* \frac{a_* \sin \theta_*}{a} [(\frac{\cot \theta_*}{a} - \frac{z_* \cos^3 \theta_*}{2a \sin \theta_*} + \frac{\lambda \varphi_* a_*}{2a})(\frac{\cos \theta_*}{\sin^2 \theta_*} + \\ & + \lambda \dot{\varphi}_* \frac{\sin \theta_*}{\cos^2 \theta_*} - \frac{\lambda_* \dot{\varphi}_*}{\cos \theta_*} + (\frac{1 - \lambda \varphi_* \tan \theta_*}{\sin \theta_*})[\dot{\eta}_* + \\ & + \frac{a_*}{a} (1 - \lambda \varphi \tan \theta_*)] \end{aligned} \quad (\text{A5})$$

3. Eqs. (24) and (27) yield at DP:

$$\chi = \tan^2 \theta_* - \frac{1}{a} (\sum_{n=0}^{\infty} \frac{a_n}{n} + \sum_{n=0}^{\infty} \frac{b_n}{n+1})$$

and, for $n=3$ in the sum of a_n 's remembering (A3):

$$\chi = \tan^2 \theta_* - \frac{1}{\alpha} \left(\frac{1}{6} \ddot{z}_* - \frac{5}{6} \dot{z}_* + \frac{11}{6} z_* \right) + \frac{11}{6\alpha} z_\infty \quad (\text{A6})$$

Since

$$z_* = \frac{hr_*}{w_*} = \frac{a_* \sin \theta_*}{n_* \cos^2 \theta_*} ; \quad z_\infty = \frac{1}{\kappa} = \frac{\chi + 1}{\epsilon + 1}$$

Eq. (A6) can be written:

$$\chi - \frac{11}{6\alpha} \frac{\chi + 1}{\epsilon + 1} = F_1(\theta_*, n_*) \quad (\text{A7})$$

where:

$$F_1(\theta_*, n_*) = \tan^2 \theta_* - \frac{1}{6\alpha} (\ddot{z}_* - 5\dot{z}_* + 11z_*) \quad (\text{A8})$$

is a function depending only on quantities at DP.

In a similar way, from the second of Eqs. (1), by integrating from DP up to infinity (and neglecting $L/D \tan \varphi$, and variations of k):

$$\log \frac{w_\infty}{w_*^2} = \frac{2}{3} - \frac{1}{6} \frac{1}{\alpha_* \tan \theta_*} \frac{2n_*}{a_* \sin \theta_* \sqrt{1 + (L/D)_*^2}} + \frac{1}{3} \frac{\sin \theta_*}{\sin \theta_*} - \frac{\cos^2 \theta_*}{\alpha_*} \sum_{i=1}^{\infty} \frac{b_n}{n-1} \quad (\text{A9})$$

where θ_e is the lowest height at which $\theta = \theta_e$. By letting $\sin \theta_e \simeq \tan \theta_e$:

$$\sin \theta_e \simeq \tan \theta_e = \sqrt{\frac{\chi+1}{\epsilon+1} \frac{\epsilon+1-\xi_e}{(1-\xi_e)^2} - 1} \quad (\text{A10})$$

whence

$$-\log \frac{\chi+1}{\epsilon+1} - \frac{1}{3} \frac{\sin \theta_e}{\sqrt{\frac{\chi+1}{\epsilon+1} \frac{\epsilon+1-\xi_e}{(1-\xi_e)^2} - 1}} = F_2(n_*, \theta_*)$$

with

$$F_2 = \frac{2}{3} - \frac{1}{6} \frac{1 - \frac{2n_*}{a_* \sin \theta_* \sqrt{1 + (L/D)_*^2}}}{a_* \tan \theta_*} - \frac{\cos^2 \theta_*}{a_*} \sum_{i^n} \frac{b_n}{n-1} - \log z_* \quad (\text{A11})$$

Eqs. (A7)(A11) are the same as Eqs. (28)(29) of Part I.

GLOSSARY OF SYMBOLS

a_n	- normal acceleration (in Part I, coefficient of Eq. (26))
a_t	- tangent acceleration
b_n	- coefficients of expansion (26)
c_D	- drag coefficient
g	- gravity acceleration
h	- gravitational constant
\dot{k}	- $(c_D A)/m$
m	- body mass
n	- ratio of deceleration to gravity
q	- heat rate
r	- radius from earth center
	- time
w	- areal velocity
x	- nondimensional density
z	- altitude
A	- main cross-area
C_p	- nondimensional value of generic P -property
D	- drag
E	- total energy
L	- lift
P	- point of trajectory
Q	- total heat transferred
V	- velocity

α	- planetary atmosphere constant
β	- α/R
ϵ	- nondimensional total energy
φ	- function defining lift modulation (Eq.(5))
η	- nondimensional k (Eq.(5))
κ	- limiting conic parameter
λ	- parameter of lift modulation (Eq.(5))
θ	- flight path angle
ψ	- rotation of flight plane
ρ	- air density
ξ	- nondimensional altitude (Eq.(4))
χ	- nondimensional conic parameter
Δa_n	} components of perturbing acceleration on normal, tangent, binormal, respectively
Δa_t	
Δa_b	

SUBSCRIPTS

*	- denotes values at DP
∞	- values at infinity
1	- reference values
C	- values of limiting conic
f	- final values

R E F E R E N C E S

1 . Luigi Broglio

AN EXACT SIMILARITY LAW AND A METHOD OF INTEGRATION FOR RE-ENTRY TRAJECTORIES

SIARgraph No.61. Contract No.AF-61(052) 198. TN1. September 1961. Rome.

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